60 deg to approximately -0.01 for the long trusses. The variation is explained by considering the two contributions to beam deflection, bending, and shear. Equivalent beam bending in a truss is due primarily to compression and tension of the truss members, while equivalent shear is due primarily to the relative motion of the adjacent bays. Since tension and compression result in little relative motion of the bays, the equivalent shear is the primary contributor to energy dissipation. Just as shear deflection becomes a small percentage of the total deflection as a beam is lengthened, the level of damping decreases with truss length.

Conclusions

Displacement dependent friction can be used to model the damping in several common types of truss joints in which the frictional forces are due to elastic deflection rather than mechanical preloads. In sleeve-stiffened beam joints, a maximum friction damping is obtained when the relative rotational stiffness of the joint and beam are of the same order. For pin joints in multielement trusses, a maximum frictional damping occurs for trusses of low length/bay-depth ratio, and large pin-radius/bay-depth ratio. The losses increase as the coefficient of friction is increased.

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Bending Eigenfrequencies of a Two-Bar Frame Including the Effect of Axial Inertia

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Introduction

WHEN considering free lateral vibrations of slender bars and framed structures, the axial force due to the longitudinal inertia is often disregarded within the scope of the linear dynamic analysis. Hohenemser and Prager[1,2] have investigated the influence of the longitudinal inertia upon the natural frequencies of free lateral vibrations. They have shown that the foregoing simplification is reasonable for frames consisting of bars with slenderness ratios greater than 40; for bars with smaller slenderness ratios, the omission of this effect may lead to errors as high as 10%.

This paper reconsiders the aforementioned effect in case of the free lateral vibrations of frames, including the translational inertia of the mass of their joints. To this end, a two-bar frame with various bar slenderness ratios, moments of inertia, and length ratios is considered. To the knowledge of the author, the effect of the longitudinal inertia of a concentrated mass upon the flexural eigenfrequencies has been discussed only for the case of dynamic buckling of a simply supported beam.[3]

Received Feb. 20, 1985; revision received May 7, 1985.

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Formulation of the Problem

We consider the free lateral motion of the uniform two-bar frame shown in Fig. 1, supported on two immovable hinges, whose joint mass is equal to M. Each bar is of length ℓ_i, mass per unit length m_i, cross-sectional area A_i, and moment of inertia $I_i\,(i=1,2)$. Let $w_i(x)$ and $\xi_i(x)$ be the lateral and axial displacement components referring to the centerline of the ith bar.

The differential equations governing the motion of the frame are

$$\left.\begin{aligned} &EI_i w_i''''(x_i,t)+m_i\ddot{w}_i(x_i,t)=0\\ &EA\xi_i''(x_i,t)-m_i\ddot{\xi}_i(x_i,t)=0 \end{aligned}\right\}\quad (i=1,2) \tag{1}$$

The associated boundary conditions are

$$\begin{gathered} w_i(0,t)=0,\quad \xi_i(0,t)=0 \quad (i=1,2)\\ w_1'(\ell_1,t)=w_2'(\ell_2,t)\\ w_1(\ell_1,t)=\xi_2(\ell_2,t),\quad w_2(\ell_2,t)=-\xi_1(\ell_1,t)\\ w_i''(0,t)=0 \quad (i=1,2)\\ EI_1w_1'''(\ell_1,t)-EA_2\xi_2'(\ell_2,t)-M\ddot{w}_1(\ell_1,t)=0\\ EI_2w_2'''(\ell_2,t)+EA_1\xi_1'(\ell_1,t)-M\ddot{w}_2(\ell_2,t)=0\\ EI_1w_1''(\ell_1,t)+EI_2w_2''(\ell_2,t)=0 \end{gathered} \tag{2}$$

For a free motion, one may assume

$$w_i(x_i,t)=W_i(x_i)e^{i\omega t},\quad \xi_i(x_i,t)=\Xi_i(x_i)e^{i\omega t} \tag{3}$$

where $i=\sqrt{-1}$ and ω is the circular frequency of the motion.

To facilitate the solution, we introduce the following dimensionless quantities:

$$\begin{gathered} \bar{x}_i=x_i/\ell_i,\quad \bar{w}_i=W_i/\ell_i,\quad \bar{\xi}_i=\Xi_i/\ell_i\\ k_i^4=m_i\ell_i^4\omega^2/EI_i,\quad \lambda_i^2=A_i\ell_i^2/I_i,\quad \upsilon_i^4=k_i^4/\lambda_i^2\\ \mu=I_2/I_1,\quad \rho=\ell_2/\ell_1,\quad \bar{M}=M/m_1\ell_1 \quad (i=1,2) \end{gathered} \tag{4}$$

By means of Eq. (4), Eqs. (1) and (2) become

$$\bar{w}_i''''-k_i^4\bar{w}_i=0 \text{ and } \bar{\xi}_i''+\upsilon_i^4\bar{\xi}=0 \quad (i=1,2) \tag{5}$$

and

$$\begin{gathered} \bar{w}_i(0)=0,\quad \bar{\xi}_i(0)=0 \quad (i=1,2)\\ \bar{w}_1'(1)=\bar{w}_2'(1),\quad \bar{w}_1(1)=\rho\bar{\xi}_2(1),\quad \bar{w}_2(1)=-(1/\rho)\bar{\xi}_1(1)\\ \bar{w}_i''(0)=0 \quad (i=1,2)\\ w_1'''(1)-(\mu/\rho^2)\lambda_2^2\bar{\xi}_2(1)+\bar{M}k_1^4\bar{w}_1(1)=0\\ \bar{w}_2'''(1)+(\rho^2/\mu)\lambda_1^2\bar{\xi}_1'(1)+(\bar{M}\rho^3/\mu)k_1^4\bar{w}_2(1)=0\\ \rho\bar{w}_1''(1)+\mu\bar{w}_2''(1)=0 \end{gathered} \tag{6}$$

Integration of Eqs. (5) and use of the boundary conditions $\bar{w}_i(0)=\bar{\xi}_i(0)=\bar{w}_i''(0)=0$ $(i=1,2)$ yield

$$\bar{w}_i(\bar{x}_i)=B_i\sin k_i\bar{x}_i+C_i\sinh k_i\bar{x}_i \qquad (i=1,2)$$

$$\bar{\xi}_i(\bar{x}_i)=D_i\sin v_i\bar{x}_i \qquad (i=1,2) \tag{7}$$

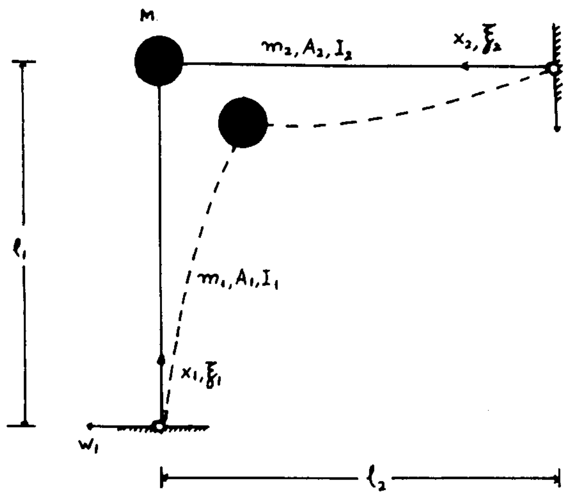

Fig. 1 Geometry and sign convention.

Table 1 Nondimensionalized eigenfrequencies k_1 for $\bar{M}=0$, $\rho=5$, and various values of μ,λ_1,λ_2

$\rho=\ell_2/\ell_1$	$\mu=I_2/I_1$	$\lambda_1=\lambda_2=30$	$\lambda_1=\lambda_2=80$	$\lambda_1=\lambda_2=150$
5	0.1	1.21457 (1.74398)	1.73359 (1.74398)	1.74265 (1.74398)
5	0.5	1.62973 (1.70331)	1.69973 (1.70331)	1.70238 (1.70331)
5	1	1.63669 (1.66469)	1.66205 (1.66469)	1.66397 (1.66469)
5	5	1.52799 (1.53383)	1.53305 (1.53383)	1.53361 (1.53383)
5	10	1.48143 (1.48438)	1.48395 (1.48438)	1.48427 (1.48438)

where B_i, C_i, and $D_i(i=1,2)$ are integration constants associated with the remaining boundary conditions that lead to a linear homogeneous system. For a nontrivial solution, the determinant of the coefficient matrix $|\alpha_{ij}|$ $(i,j=1,2,3,4)$ must be zero, i.e.,

$$|\alpha_{ij}|=0 \tag{8}$$

where

$$\alpha_{11}=k_1\cos k_1, \quad \alpha_{12}=k_1\cosh k_1,$$

$$\alpha_{13}=-k_2\cos k_2, \quad \alpha_{14}=-k_2\cosh k_2$$

$$\alpha_{21}=-\frac{k_1^3}{\lambda_2^2}\cos k_1-\frac{\mu v_2}{\rho^3}\sin k_1\frac{\cos v_2}{\sin v_2}+\frac{\bar{M}k_1^4}{\lambda_2^2}\sin k_1$$

$$\alpha_{22}=\frac{k_1^3}{\lambda_2^2}\cosh k_1-\frac{\mu v_2}{\rho^3}\sinh k_1\frac{\cos v_2}{\sin v_2}+\frac{\bar{M}k_1^4}{\lambda_2^2}\sinh k_1$$

$$\alpha_{23}=\alpha_{24}=0, \quad \alpha_{31}=\alpha_{32}=0$$

$$\alpha_{33}=-\frac{k_2^3}{\lambda_1^2}\cos k_2-\frac{\rho^3 v_1}{\mu}\sin k_2\frac{\cos v_1}{\sin v_1}+\frac{\mu}{\lambda_1^2}k_1^4\frac{\rho^3}{\mu}\sin k_2$$

$$\alpha_{34}=\frac{k_2^3}{\lambda_1^2}\cosh k_2-\frac{\rho^3}{\mu}v_1\sinh k_2\frac{\cos v_1}{\sin v_1}+\frac{\bar{\mu}}{\lambda_1^2}k_1^4\frac{\rho^3}{\mu}\sinh k_2$$

$$\alpha_{41}=-k_1^2\sin k_1, \quad \alpha_{42}=k_1^2\sinh k_1$$

$$\alpha_{43}=-\frac{\mu}{\rho}k_2^2\sin k_2, \quad \alpha_{44}=\frac{\mu}{\rho}k_2^2\sinh k_2$$

The vanishing of the determinant leads to the frequency equation of lateral vibrations of the frame including the effect of axial motion. Clearly, for λ_1, $\lambda_2\to\infty$, this effect is neglected. Given that

$$v_1^4=k_1^4/\lambda_1^2, \quad v_2^4=\rho^2k_1^4/\lambda_1^2, \quad k_2=k_1\sqrt{(\lambda_2/\lambda_1)\rho} \tag{9}$$

the frequency equation (8) is a transcendental equation of the form

$$f(k_1,\rho,\mu,\lambda_1,\lambda_2,\bar{M})=0 \tag{10}$$

Numerical Results and Discussion

The nondimensionalized eigenfrequency k_1 is established as function of the parameters ρ, μ, λ_1, λ_2, and $\bar{M}$ by solving Eq. (8) numerically. Numerical results showing the individual and coupling effect on the fundamental frequency

Table 2 Nondimensionalized bending eigenfrequencies k_1 for $\bar{M}=0$, 1, 10 and various values of ρ, μ, λ_1, and λ_2 (numbers in parentheses correspond to eigenfrequencies without the effect of axial motion)

		$M=0$			$M=1$			$M=10$		
$\rho=\ell_2/\ell_1$	$\mu=I_2/I_1$	$\lambda_1=\lambda_2=30$	$\lambda_1=\lambda_2=80$	$\lambda_1=\lambda_2=150$	$\lambda_1=\lambda_2=30$	$\lambda_1=\lambda_2=80$	$\lambda_1=\lambda_2=150$	$\lambda_1=\lambda_2=30$	$\lambda_1=\lambda_2=80$	$\lambda_1=\lambda_2=150$
1	0.1	2.97509 (3.14166)	3.11929 (3.14166)	3.13531 (3.14166)	2.72722 (3.14166)	3.11533 (3.14166)	3.13506 (3.14166)	1.71650 (3.14166)	2.78226 (3.14166)	3.13060 (3.14166)
	1	3.12375 (3.14166)	3.13921 (3.14166)	3.14096 (3.14166)	3.12155 (3.14166)	3.13914 (3.14166)	3.14090 (3.14166)	2.97433 (3.14166)	3.13677 (3.14166)	3.14090 (3.14166)
	10	2.97509 (3.14166)	3.11929 (3.14166)	3.13531 (3.14166)	2.95932 (3.14166)	3.11891 (3.14166)	3.13531 (3.14166)	2.72722 (3.14166)	3.11533 (3.14166)	3.13506 (3.14166)
10	0.1	0.75951 (1.23760)	1.13902 (1.23760)	1.23044 (1.23760)	0.54494 (1.23760)	0.83749 (1.23760)	1.13085 (1.23760)	0.32792 (1.23760)	0.50429 (1.23760)	0.68493 (1.23760)
	1	1.09522 (1.20581)	1.19211 (1.20581)	1.20204 (1.20581)	0.92438 (1.20581)	1.18551 (1.20581)	1.20166 (1.20581)	0.57755 (1.20581)	0.89096 (1.20581)	1.17759 (1.20581)
	10	1.06971 (1.08617)	1.08372 (1.08617)	1.08548 (1.08617)	1.06726 (1.08617)	1.08366 (1.08617)	1.08548 (1.08617)	0.96824 (1.08617)	1.08303 (1.08617)	1.08542 (1.08617)

of the cross-sectional axial inertia and the concentrated joint mass $\bar{M}$ are presented in Tables 1 and 2.

The fundamental frequencies are evaluated for three characteristic cases ($\bar{M}=0$, 1, and 10) for frames with $\rho=1$, 5, 10 and $\mu=0.1$, 1, and 10 and three different sets of slenderness ratios ($\lambda_1=\lambda_2=30$, 80, and 150). The eigenfrequencies without the effect of axial motion (λ_1, $\lambda_2 \rightarrow \infty$) are shown in parentheses in Tables 1 and 2. From all of the cases considered, it is clear that the eigenfrequencies of the frame decrease when the effect of axial motion is taken into account.

From Table 1, one can observe the effect of axial inertia on the bending eigenfrequency for frames with $\bar{M}=0$, $\rho=5$ and various values of stiffness and slenderness ratios. For the case $\lambda_1=\lambda_2=30$ and $\mu=0.1$ (corresponding to $A_2=0.004\ A_1$), the aforementioned effect decreases the eigenfrequency by 43.5%. The case of frames without joint mass ($\bar{M}=0$) is also included in Table 2 for the same sets of slenderness ratios having the parameter values $\rho=1$ and 10 and $\mu=0.1$, 1, and 10. For the extreme case in which $\rho=10$, $\mu=0.1$ (corresponding to $A_2=0.001\ A_1$), and $\lambda_1=\lambda_2=30$, the foregoing effect diminishes the eigenfrequency by about 63%. The conclusion drawn from the above discussion, associated with $\bar{M}=0$, is that the effect of axial inertia becomes considerable when a high length ratio is combined with low moments of inertia and slenderness ratios. For stubby beams, one should also include the effects of the transverse shear and rotary inertia[4,5] of the cross section.

Table 2 also includes the cases $\bar{M}=1$ and 10. From this table it is clear that, due to the presence of the joint mass $\bar{M}$, the effect of axial inertia on the fundamental eigenfrequency becomes more pronounced.

From the above results, it is deduced that the effect of longitudinal motion on the bending fundamental eigenfrequency might be considerable if the joint mass is taken into account, particularly in case of bars with low slenderness ratios. Even in the case in which the mass of the joint is not taken into account, the foregoing effect might be appreciable under certain combinations of length and stiffness ratios.

Conclusions

From this investigation, one can draw the following conclusions:

1) The effect of axial inertia on the lateral vibrations of a two-bar frame can decrease its fundamental eigenfrequency considerably if the mass of the joint is taken into account.

2) If the aforementioned joint mass is neglected, this effect diminishes the bending eigenfrequencies; however, it might be appreciable only for frames in which a high length ratio is combined with a low moment of inertia and low slenderness ratios.

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